1) найдем время подъема над поверхностью до максимальной высоты

За пределами тоннеля $R \le r \le 2R$ тело движется с ускорением $r`` = -g*\left(\frac{R}{r}\right)^2$

При r=2R r`=0

$$r`` = -g*\left(\frac{R}{r}\right)^2 \Rightarrow r``r^2 = -g*R^2$$

$$\frac{dr}{dt} = r` = v = v(r) \Rightarrow r`` = \frac{dv(r)}{dt} = \frac{dv(r)}{dr}*\frac{dr}{dt} = \frac{dv(r)}{dr}*v \Rightarrow$$

$$\frac{dv(r)}{dr}*v*r^2 = -g*R^2 \Rightarrow v*dv = -gR^2*\frac{dr}{r^2} \Rightarrow \frac{v^2}{2} = gR^2*\frac{1}{r}+C$$

$$v = 0 \ при \ r = 2R \Rightarrow \frac{v^2}{2} = gR^2*\frac{1}{r} - gR^2*\frac{1}{2R} = gR^2*\left(\frac{1}{r}-\frac{1}{2R}\right) \Rightarrow |v| = \sqrt{2gR^2}*\sqrt{\frac{1}{r}-\frac{1}{2R}}$$

$$\frac{dr}{dt} = \sqrt{2gR^2}*\sqrt{\frac{1}{r}-\frac{1}{2R}} \Rightarrow \frac{dr}{\sqrt{\frac{1}{r}-\frac{1}{2R}}} = \sqrt{2gR^2}*dt$$

$$\sqrt{\frac{1}{r}-\frac{1}{2R}} = y \Rightarrow \frac{1}{r} = y^2+\frac{1}{2R} \Rightarrow r = \frac{1}{y^2+\frac{1}{2R}} \Rightarrow dr = -\frac{2ydy}{\left(y^2+\frac{1}{2R}\right)^2}$$

$$r \in (R;2R) \Rightarrow y \in \left(\sqrt{\frac{1}{R}-\frac{1}{2R}};\sqrt{\frac{1}{2R}-\frac{1}{2R}}\right) \equiv \left(\sqrt{\frac{1}{2R}};0\right)$$

$$\int\frac{dr}{\sqrt{\frac{1}{r}-\frac{1}{2R}}} = -\int\frac{2ydy}{\left(y^2+\frac{1}{2R}\right)^2}\frac{1}{y} = -\int\frac{2dy}{\left(y^2+\frac{1}{2R}\right)^2} = \left\{\begin{array}{l} y = \frac{1}{\sqrt{2R}}tg\varphi; dy = \frac{1}{\sqrt{2R}}*\frac{d\varphi}{\cos^2\varphi} \\ y^2+\frac{1}{2R} = \frac{1}{2R}\left(1+tg^2\varphi\right) = \frac{1}{2R}\frac{1}{\cos^2\varphi} \\ y \in \left(\sqrt{\frac{1}{2R}};0\right) \Rightarrow \varphi \in \left(\frac{\pi}{4};0\right) \end{array}\right\} =$$

$$= -\int\frac{2\frac{1}{\sqrt{2R}}*\frac{d\varphi}{\cos^2\varphi}}{\left(\frac{1}{2R}\frac{1}{\cos^2\varphi}\right)^2} = -2\frac{(2R)^2}{\sqrt{2R}}\int\cos^2\varphi*d\varphi = -2(2R)^{\frac{3}{2}}\int\frac{\cos(2\varphi)+1}{2}*d\varphi =$$

$$= -2(2R)^{\frac{3}{2}}\left(\frac{\varphi}{2}+\frac{\sin 2\varphi}{4}\right) = \sqrt{2gR^2}*\int dt = \sqrt{2gR^2}*t+C$$

$$\Delta t = -\frac{2(2R)^{\frac{3}{2}}}{\sqrt{2gR^2}}\left(\frac{\varphi}{2}+\frac{\sin 2\varphi}{4}\right)_{\varphi=\frac{\pi}{4}}^{0} = \frac{2(2R)^{\frac{3}{2}}}{\sqrt{2gR^2}}\left(\frac{\frac{\pi}{4}}{2}+\frac{\sin\frac{\pi}{2}}{4}\right) = \frac{(2R)^{\frac{3}{2}}}{\sqrt{2gR^2}}\left(\frac{\pi}{4}+\frac{1}{2}\right) = \sqrt{\frac{R}{g}}\left(\frac{\pi+2}{2}\right)$$

$$\Delta t = \sqrt{\frac{R}{g}}\left(\frac{\pi+2}{2}\right) - искомое\ время$$

$|v|_{r=R} = \sqrt{2gR^2}*\sqrt{\frac{1}{R}-\frac{1}{2R}} = \sqrt{2gR^2}*\sqrt{\frac{1}{2R}} = \sqrt{gR}$ - скорость у поверхности

2) найдем время движения в тоннеле в одном направлении от центра до поверхности

$$r`` = -g\frac{r}{R} \Rightarrow r = A*\sin\left(\sqrt{\frac{g}{R}}*t\right) \Rightarrow r` = A*\sqrt{\frac{g}{R}}*\cos\left(\sqrt{\frac{g}{R}}*t\right)$$

$$r(t=t_1) = A*\sin\left(\sqrt{\frac{g}{R}}*t_1\right) = R$$

$$r`(t=t_1) = A*\sqrt{\frac{g}{R}}*\cos\left(\sqrt{\frac{g}{R}}*t_1\right) = \sqrt{gR}$$

$$\begin{cases} A*\sin\left(\sqrt{\frac{g}{R}}*t_1\right) = R \\ A*\cos\left(\sqrt{\frac{g}{R}}*t_1\right) = \frac{\sqrt{gR}}{\sqrt{\frac{g}{R}}} = R \end{cases} \Rightarrow tg\left(\sqrt{\frac{g}{R}}*t_1\right) = 1 \Rightarrow \sqrt{\frac{g}{R}}*t_1 = \frac{\pi}{4} \Rightarrow t_1 = \frac{\pi}{4}\sqrt{\frac{R}{g}}$$

3) финальный этап

$$g = \frac{GM}{R^2} = \frac{GV\rho}{R^2} = \frac{G4\pi\rho R^3}{3R^2} = \frac{4G\pi\rho}{3}R$$

$$T = 4*(t_1+\Delta t) = 4*\left(\frac{\pi}{4}\sqrt{\frac{R}{g}} + \sqrt{\frac{R}{g}}\left(\frac{\pi+2}{2}\right)\right) = \sqrt{\frac{R}{g}}(\pi+2(\pi+2)) = \sqrt{\frac{R}{g}}(3\pi+4) =$$

$$= \sqrt{\frac{R}{\frac{4G\pi\rho}{3}R}}(3\pi+4) = \sqrt{\frac{3}{4G\pi\rho}}(3\pi+4) = 11715{,}23\ \text{сек} \sim 195\ \text{мин} = 3\text{ч}\,15\text{мин}$$

T ~ 3 часа 15 мин – это ответ